International Journal of Theoretical Physics, Vol. 8, No. 1 (1973), pp. 11–29

On the Isomorphism of a 'Quantum Logic' with the Logic of the Projections in a Hilbert Space

R. CIRELLI

Istituto di Scienze Fisiche dell'Università, Milano, Istituto Nazionale de Fisica Nucleare, Sezione di Milano

and

P. COTTA-RAMUSINO†

Istituto di Scienze Fisiche dell'Università, Milano

Received: 10 *July* 1972

Abstract

A topology is introduced in a logic $\mathscr{L}$ using the set of pure states of $\mathscr{L}$. It is shown that $\mathscr{L}$, equipped with this topology, under suitable conditions, determines the division ring $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$. With the continuity of the antiautomorphism of the division ring added, it is shown that these conditions are necessary and sufficient for the projective logic $\mathscr{L}$ to be isomorphic with the projective logic of the projections in a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$.

1. *Introduction*

In the framework of the axiomatic approach to the foundations of quantum mechanics which is known as the 'quantum logic approach', one meets with the problem of the conditions under which a quantum logic admits a 'hilbertian representation'. In this connection the main result is the Piron theorem (Piron, 1964), which we quote in the formulation given by Varadarajan (1968):

> 'Let $\mathscr{L}$ be any logic. Then necessary and sufficient condition that $\mathscr{L}$ be isomorphic to the logic of all closed linear manifolds of a separable Hilbert space over the division ring $\mathbb{D}$ (which is one of $\mathbb{R}$, the real field, $\mathbb{C}$, the complex field, or $\mathbb{Q}$, the quaternion division ring) is that $\mathscr{L}$ be a projective logic *associated with* $\mathbb{D}$ and have the property that every family of mutually orthogonal points of $\mathscr{L}$ must be countable'.

† P. Cotta-Ramusino gratefully acknowledges a fellowship of the Consiglio Nazionale delle Ricerche.

In Section 2 we will see what is meant by a *logic associated with* $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$). To introduce the question we examine in this paper, we point out that the conditions for a logic to be associated with $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$)—that is, the conditions under which a logic leads to a quantum mechanics over the real field, the complex field or the quaternion division ring—are left unsettled. As Varadarajan says, 'this is a classical question and is intimately connected with the topologies on a geometry'.

In this paper we introduce a topology in a logic, which on one side is interesting from the physical point of view, and on the other allows us to say that the division ring associated with the logic is really $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$.

In Section 2 we recall some useful definitions contained in Varadarajan's book. In Section 3 we study some topological properties of the lattice of the projections in a separable Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$. In Section 4 we introduce in a logic the 'topology of states', and in Section 5 we examine the conditions under which a logic with the topology of states determines the division ring $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$.

The mathematical tools that one must know to read the paper are collected in the Appendix, in which will be found all the definitions and notations not given in the text.

2. *Basic Concepts*

(a) A *logic* is a lattice with $\mathbf{0}$ and $\mathbf{I}$, σ-complete, ortho-complemented and weakly modular.

(b) An *observable* associated with a logic $\mathscr{L}$ is a mapping $\alpha\colon B(\mathbb{R}) \to \mathscr{L}$ [where $B(\mathbb{R})$ is the σ-algebra of Borel sets of the real line] such that:

(1) $\alpha(\phi) = \mathbf{0}, \quad \alpha(\mathbb{R}) = \mathbf{I}$;
(2) $E, F \in B(\mathbb{R}), \quad E \cap F = \phi \Rightarrow \alpha(E) \leqslant (\alpha(F))^{\perp}$
(3) $\{E_n\}_{n=1,2,\ldots} \subset B(\mathbb{R}), \quad E_i \cap E_j = \phi \quad \text{for } i \neq j \to \alpha(\bigcup E_n) = \vee \alpha(E_n)$

If α is an observable and f a Borel real function on $\mathbb{R}$ we define the observable $f \circ \alpha$ as the mapping given by $(f \circ \alpha)(E) =: \alpha(f^{-1}(E))\ \forall\ E \in B(\mathbb{R})$

(c) If $\mathscr{O}$ is the set of all observables associated with a logic $\mathscr{L}$ and π is the set of all the probability distributions on $B(\mathbb{R})$, a *state function* of $\mathscr{L}$ is a mapping $\rho\colon \mathscr{O} \to \pi$ such that $(\rho(f \circ \alpha))(E) = (\rho(\alpha))(f^{-1}(E))$, $\forall\ E \in B(\mathbb{R})$ for every $\alpha \in \mathscr{O}$ and every Borel real function f on $\mathbb{R}$.

(d) A *state* associated with a logic $\mathscr{L}$ is a mapping $s\colon \mathscr{L} \to \mathbb{R}$ such that:

(1) $0 \leqslant s(a) \leqslant 1, \quad \forall\ a \in \mathscr{L}$;
(2) $s(\mathbf{0}) = 0, \quad s(\mathbf{I}) = 1$;
(3) $\{a_n\}_{n=1,2,\ldots} \subset \mathscr{L},\ a_i \leqslant a_j^{\perp} \quad \text{for } i \neq j \Rightarrow s(\vee a_n) = \sum_n s(a_n)$.

It can be shown (Varadarajan, 1968, Theorem 6.5) that, given a state s, for every observable $\alpha \in \mathscr{O}$, $(\rho^s(\alpha))(E) =: s(\alpha(E))$ defines a probability distribution on $B(\mathbb{R})$ and that the mapping $\alpha \rightsquigarrow \rho^s(\alpha)$ is a state function of $\mathscr{L}$. Conversely, given a state function ρ there exists one and only one state s such that, for every $\alpha \in \mathscr{O}$, $(\rho(\alpha))(E) = s(\alpha(E)), \forall\ E \in B(\mathbb{R})$.

A state s is called a *pure state* if $s = ts_1 + (1-t)s_2$ with $0 \leqslant t \leqslant 1$ and s_1, s_2 states, implies $s_1 = s_2 = s$.

In his definition of a state Jauch (1968) requires, besides conditions (1), (2) and (3), the following ones also:

(4) $s(a_i) = 1$ for every i belonging to an at most countable set of indices implies $s(\wedge\, a_i) = 1$;
(5) the set of states is separate.

In the following we shall call $\mathscr{S}$ the set of all states of $\mathscr{L}$ satisfying also (4) and (5) and $\mathscr{P}$ the set of all states of $\mathscr{S}$ which are pure. Conditions (4) and (5) are convenient from the point of view of the physical interpretation; however their omission would not cause us any serious mathematical difficulty.

(e) A *projective logic* is a logic $\mathscr{L}$ such that:

(1) $\mathscr{L}$ is atomic;
(2) if $a \in \mathscr{L}$, $a \neq \mathbf{0}$ is the lattice union of a finite number of atoms (such an a is called a *finite* element of $\mathscr{L}$), then $\mathscr{L}[\mathbf{0}, a]$ is a geometry;
(3) if $a \in \mathscr{L}$, $a \neq \mathbf{0}$, $a \neq \mathbf{I}$ and p is an atom of $\mathscr{L}$, then there exist in $\mathscr{L}$ two atoms q and r such that $q \leqslant a, \quad r \leqslant a^{\perp}, \quad p \leqslant q \vee r$;
(4) there exists in $\mathscr{L}$ a finite element a such that $\dim \mathscr{L}[\mathbf{0}, a] \geqslant 4$

Since for every finite element a of $\mathscr{L}$, $\mathscr{L}[\mathbf{0}, a]$ is a geometry, we can define $\dim(a)$ for each finite element a setting $\dim(a) =: \dim \mathscr{L}\ [\mathbf{0}, a]$. Obviously, for a point p, $\dim(p) = 1$; a finite element a with $\dim(a) = 2$ will be called a *line* of $\mathscr{L}$, a finite a with $\dim(a) = 3$ will be called a *plane* of $\mathscr{L}$, and so on.

The lattice $\mathscr{L}(\mathscr{H}, \mathbb{D})$ of all the projections of a Hilbert space over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with $\dim \mathscr{H} \geqslant 4$ is a projective logic. The finite elements of $\mathscr{L}(\mathscr{H}, \mathbb{D})$ are the projections of finite rank; obviously, the atoms are the monodimensional projections, the lines are the bidimensional projections and the planes are the tridimensional projections.

Now let V be a linear space over a division ring $\mathbb{K}$ with $\dim V \geqslant 4$, θ an involutive antiatomorphism of $\mathbb{K}$, and $\langle .,. \rangle$ a definite symmetric θ-bilinear form on VxV. For every subset M of V we define M^* as

$$M^* =: \{x \in V \mid \langle \mu, x \rangle = 0, \quad \forall \mu \in M\}$$

Obviously M^* is a linear manifold of V. We will say that a linear manifold N of V is closed relative to $\langle .,. \rangle$, or $\langle .,. \rangle$-closed, if $N = N^{**}$. Moreover we will say that the pair $(V, \langle .,. \rangle)$ is hilbertian if, for every $\langle .,. \rangle$-closed linear manifold N of V, $V = N + N^*$.

Then the following theorem can be proved (Varadarajan, 1968, Theorem 7.40:

'Let $\mathbb{K}$ be a division ring, V a linear space over $\mathbb{K}$ with $4 \leqslant \dim V$, θ an involutive antiautomorphism of $\mathbb{K}$ and $\langle .,. \rangle$ a definite symmetric θ-bilinear form on VxV. Let $\mathscr{L}(V, \langle .,. \rangle)$ be the set of all $\langle .,. \rangle$-closed linear manifolds of V partially ordered under inclusion. If $(V, \langle .,. \rangle)$ is hilbertian then $\mathscr{L}(V,\langle.,.\rangle)$ is a complete projective logic, and for any collection $\{M_j\}$ of $\langle.,.\rangle$-closed linear manifolds of V the lattice operations in $\mathscr{L}(V,\langle.,.\rangle)$ are given by

$$\bigvee_j \{M_j\} = (\bigcup M_j)^{**}, \qquad \bigwedge_j \{M_j\} = \bigcap_j M_j$$

Conversely, let $\mathscr{L}$ be any complete projective logic. Then there exists a division ring $\mathbb{K}$, an involutive antiautomorphism θ of $\mathbb{K}$ a vector space V over $\mathbb{K}$ and a definite symmetric θ-bilinear form $\langle.,.\rangle$ on VxV such that $(V, \langle.,.\rangle)$ is hilbertian and $\mathscr{L}$ is isomorphic to $\mathscr{L}(V,\langle.,.\rangle)$.'

The division ring $\mathbb{K}$ of the second part of the theorem just quoted is *uniquely determined* by $\mathscr{L}$ up to isomorphism. If this ring $\mathbb{K}$ is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, the antiautomorphism θ is uniquely determined also, and is bound to be the identity or the canonical conjugation in the case of $\mathbb{R}$ or $\mathbb{Q}$, respectively, while it does not need to be the complex conjugation in the case of $\mathbb{C}$. If the division ring $\mathbb{K}$ results in $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) and, in the case of $\mathbb{C}$, the antiautomorphism θ results in the complex conjugation, then the projective logic $\mathscr{L}$ is said to be *associated with* $\mathbb{D}$.

So we have a complete understanding of the Piron theorem quoted in the Introduction.

The question is now to ascertain under which conditions a projective logic $\mathscr{L}$ is really associated with $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$).

In the following sections we shall find necessary and sufficient conditions under which a complete projective logic determines $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$.

3. *The Weak Operator Topology On* $\mathscr{L}(\mathscr{H},\mathbb{D})$

Let $\mathscr{L}(\mathscr{H}, \mathbb{D})$ be the lattice of the projections of a separable Hilbert space over the division ring $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with $\dim \mathscr{H} \geqslant 4$. As we have noted in Section 2, $\mathscr{L}(\mathscr{H},\mathbb{D})$ is a projective logic.

We consider on $\mathscr{L}(\mathscr{H},\mathbb{D})$ the topology induced by the *weak operator topology* of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$. In this way $\mathscr{L}(\mathscr{H},\mathbb{D})$ is a metrisable *second countable space* (Dixmier, 1969).

Theorem 3.1. Let Q be a projection of finite rank. Then the geometry $\mathscr{L}[\mathbf{0}, Q]$ is a compact subset of $\mathscr{L}(\mathscr{H},\mathbb{D})$.

Proof. Since $\mathscr{L}[\mathbf{0}, Q]$ is a subset of $\mathscr{B}_1(\mathscr{H})$ (the unit ball of $\mathscr{B}(\mathscr{H})$) and $\mathscr{B}_1(\mathscr{H})$ is a compact metrisable space in the induced weak operator topology (Dixmier, 1969), it is enough to show that $\mathscr{L}[\mathbf{0}, Q]$ is sequentially closed in $\mathscr{B}_1(\mathscr{H})$. Let $\{P_n\} \subset \mathscr{L}[\mathbf{0}, Q]$ and

$$P_n \xrightarrow{W} A$$

Then A is bounded and

$$P_n^+ \xrightarrow{W} A^+$$

so that $A = A^+$. Moreover, for every $x \in \mathscr{H}$,

$$\|P_n x\|^2 = \sum_{k=1}^{N} (P_n x, u_z)(u_k, P_n x)$$

where $\{u_k\}_{k=1,\ldots,N}$ is a complete orthonormal system of R_Q (the range of Q) and so

$$\lim_{n\to\infty} \|P_n x\|^2 = \sum_{k=1}^{N} \lim_{n\to\infty} (P_n x, u_k)(u_k, P_n x) = \|Ax\|^2$$

thus ([5], V.1.2)

$$\|P_n x - Ax\| \underset{n\to\infty}{\longrightarrow} 0$$

for every $x \in \mathscr{H}$, that is

$$P_n \overset{s}{\to} A$$

But if

$$P_n \overset{s}{\to} A$$

then

$$P_n^2 \overset{s}{\to} A^2$$

so that $A = A^2$. Finally

$$(P_n x, x) \leqslant (Qx, x) \Rightarrow \lim_{n\to\infty} (P_n x, x) \leqslant (Qx, x) \Rightarrow (Ax, x) \leqslant (Qx, x)$$

that is, $A \in \mathscr{L}[\mathbb{0}, Q]$.

Theorem 3.2. Let l be any line of $\mathscr{L}(\mathscr{H}, \mathbb{D})$. Then the set of all points of l, but one arbitrarily chosen, is a connected set.

Proof. Let P be a bidimensional projection, Q_0 a projection such that $Q_0 \leqslant P$, $Q_0 \neq P$, $Q_0 \neq \mathbb{O}$, and $\mathscr{S}$ defined as follows:

$$\mathscr{S} =: \{Q \in \mathscr{L}(\mathscr{H}, \mathbb{D}) | Q \leqslant P, Q \neq P, Q \neq \mathbb{O}, Q \neq Q_0\}$$

We want to prove that $\mathscr{S}$ is a connected set. The set $\mathscr{R}$ of all vectors of $\mathscr{H}$ which belong to R_P but not to R_Q is either connected or (in the case $\mathbb{D} = \mathbb{R}$) the union of two connected subset $\mathscr{R}_1$ and $\mathscr{R}_2$. The mapping $x \rightsquigarrow P_x$, where $x \in \mathscr{R}$ (or $\mathscr{R}_1$) and P_x is the projection on the subspace generated by X is a surjective continuous mapping from $\mathscr{R}$ (or $\mathscr{R}_1$) onto $\mathscr{S}$. The surjectivity is immediate; for the continuity we note that, if $\{x_n\}$ is a sequence in $\mathscr{R}$ (or $\mathscr{R}_1$) such that $\|x_n - x\| \to 0$ with $x \in \mathscr{R}$ then, for every $y_1 z \in \mathscr{H}$

$$(P_{x_n} y, z) = \frac{1}{\|x_n\|^2} (y, x_n)(x_n, z) \to \frac{1}{\|x\|^2} (y, x)(x, z) = (P_x y, z)$$

that is

$$P_{x_n} \overset{w}{\to} P_x$$

The assertion follows from the fact that connectedness is invariant under continuity.

We want now to prove that in any plane of $\mathscr{L}(\mathscr{H},\mathbb{D})$ the intersection point of two lines is a continuous function of the two lines and that the union line of the two points is a continuous function of the two points. For each $x\in\mathscr{H}$, P_x will mean the projection on the subspace generated by x.

Lemma 3.3. Let $\{x_n\}$ be a sequence of normalised vectors in $\mathscr{H}$ and x a normalised vector in $\mathscr{H}$.

$$P_{x_n}\xrightarrow{w}P_x$$

iff $|(x,x_n)|^2\to 1$.

Proof. (a) *Necessity*. From

$$P_{x_n}\xrightarrow{w}P_x$$

we have, for every $y,z\in\mathscr{H}$ $(y,x_n)(x_n,z)\to(y,x)(x,z)$; hence, setting $y=z=x$

$$|(x,x_n)|^2\to|(x,x)|^2=1$$

(b) *Sufficiency*. We choose $\{y_i\}$ such that $\{x,y_1,y_2,\ldots\}$ is a complete orthonormal set in $\mathscr{H}$. Then

$$|(x_n,x)|^2+\sum_i|(x_n,y_i)|^2=1$$

so that

$$|(x_n,x)|^2\xrightarrow[n\to\infty]{}1$$

implies

$$\sum_i|(x_n,y_i)|^2\xrightarrow[n\to\infty]{}0$$

Now, for every $z\in\mathscr{H}$

$$(z,x_n)=(z,x)(x,x_n)+\sum_i(z,y_i)(y_i,x_n)$$

then

$$||(z,x_n)|-|(z,x)(x,x_n)||\leqslant|(z,x_n)-(z,x)(x,x_n)|=$$

$$=\left|\sum_i(z,y_i)(y_i,x_n\right|\leqslant\left(\sum_i|(z,y_i)|^2\right)^{1/2}\left(\sum_i|(y_i,x_n)|^2\right)^{1/2}\leqslant$$

$$\leqslant\|z\|\left(\sum_i|(y_i,x_n)|^2\right)^{1/2}\xrightarrow[n\to\infty]{}0$$

thus

$$|(z,x_n)|\to|(z,x)|,\qquad\forall z\in\mathscr{H}$$

whence

$$(P_{x_n}z,z)=(z,x_n)(x_n,z)=|(z,x_n)|^2\to|(z,x)|^2=(P_xz,z)$$

Using the polarisation principle, then we have

$$P_{x_n}\xrightarrow{W}P_x$$

Remark 3.1. The polarisation principle for (Ax,y) where $A \in \mathscr{B}(\mathscr{H})$ is very well known if $\mathscr{H}$ is over $\mathbb{R}$ or $\mathbb{C}$:
$\mathbb{D} = \mathbb{R}$, A hermitian:

$$4(Ax,y) = (A(x+y), x+y) - (A(x-y), x-y)$$

$\mathbb{D} = \mathbb{C}$:

$$4(Ax,y) = (A(x+y), x+y) - (A(x-y), x-y) + i(A(x+iy), x+iy) - i(A(x-iy), x-iy)$$

for $\mathbb{D} = \mathbb{Q}$ it reads

$$\begin{aligned} 4(Ax,y) = {} & (A(x+y), x+y) - (A(x-y), x-y) \\ & + j_1(A(x+j_1 y), x+j_1 y) - j_1(A(x-j_1 y), x-j_1 y) \\ & + j_2(A(x+j_2 y), x+j_2 y) - j_2(A(x-j_2 y), x-j_2 y) \\ & + j_3(A(x+j_3 y), x+j_3 y) - j_3(A(x-j_3 y), x-j_3 y) \end{aligned}$$

Let now $\mathscr{H}_3$ be any tridimensional subspace of $\mathscr{H}$. In Theorems 3.4 and 3.5 we consider *only* projections whose range is contained in $\mathscr{H}_3$.

Theorem 3.4. Let $\{P_n\}$ and $\{Q_n\}$ be two sequences of monodimensional projections. If

$$P_n \overset{W}{\to} P$$

$$Q_n \overset{W}{\to} Q$$

where P and Q are monodimensional projections and $P \neq Q$, then

$$P_n \vee Q_n \underset{u\to\infty}{\longrightarrow} P \vee Q$$

Proof. Since $P \neq Q$, for sufficiently large n it is $P_n \neq Q_n$. For each such large n let z_n be a normalised vector of $\mathscr{H}_3$ orthogonal to $R_{P_n} \vee_{Q_n}$ and let x and y be two normalised vectors belonging to R_P and R_Q respectively. We can write:

$$x = (x, x_n) x_n + (x, x_n{}^{\perp}) x_n{}^{\perp} + (x_1, z_n) z_n$$
$$y = (y, y_n) y_n + (y, y_n{}^{\perp}) y_n{}^{\perp} + (y, z_n) z_n$$

where $x_n(y_n)$ is a normalised vector of $R_{P_n}(R_{Q_n})$ and $x_n{}^{\perp}(y_n{}^{\perp})$ a normalised vector of $R_{P_n} \vee_{Q_n}$ such that $\{x_n(y_n), x_n{}^{\perp}(y_n{}^{\perp}), z_n\}$ is an orthonormal set in $\mathscr{H}_3$. Since

$$|(x, x_n)|^2 + |(x, x_n{}^{\perp})|^2 + |(x, z_n)|^2 = 1$$
$$|(y, y_n)|^2 + |(y, y_n{}^{\perp})|^2 + |(y, z_n)|^2 = 1$$

and

$$|(x, x_n)|^2 \to 1, |(y, y_n)|^2 \to 1$$

we have $|(x, z_n)|^2 \to 0$.

Let now z be a normalised vector of $\mathscr{H}_3$ orthogonal to $R_{P\vee Q}$. We have

$$z_n = \alpha_n x + \beta_n y + (z_n, z) z$$

where

$$\alpha_n = \frac{(z_n, x) - (z_n, y)(y, x)}{1 - |(x, y)|^2}, \qquad \beta_n = \frac{(z_n, y) - (z_n, x)(x, y)}{1 - |(x, y)|^2}$$

Obviously,

$$|\alpha_n| \leqslant \frac{2}{1 - |(x, y)|^2}, \qquad |\beta_n| \leqslant \frac{2}{1 - |(x, y)|^2}$$

and thus from

$$1 = (z_n, z_n) = \alpha_n(x, z_n) + \beta_n(y, z_n) + (z_n, z)(z, z_n)$$

we see that $|(x, z_n)| \to 0$ and $|(y, z_n)| \to 0$ imply $|(z, z_n)|^2 \to 1$, that is

$$P_{z_n} \overset{W}{\to} P_z$$

But

$$P_n \vee Q_n = \mathbb{1}_{\mathscr{H}_3} - P_{z_n} \qquad \text{and} \qquad P \vee Q = \mathbb{1}_{\mathscr{H}_3} - P_z$$

therefore,

$$P_n \vee Q_n \overset{W}{\to} P \vee Q$$

Theorem 3.5. Let $\{P_n\}$ and $\{Q_n\}$ be two sequences of bidimensional projections. If

$$P_n \overset{W}{\to} P, \qquad Q_n \overset{W}{\to} Q$$

where P and Q are bidimensional projections and $P \neq Q$, then

$$P_n \wedge Q_n \overset{W}{\to} P \wedge Q$$

Proof. From Theorem 3.4 we have

$$(\mathbb{1}_{\mathscr{H}_3} - P_n) \vee (\mathbb{1}_{\mathscr{H}_3} - Q_n) \overset{W}{\to} (\mathbb{1}_{\mathscr{H}_3} - P) \vee (\mathbb{1}_{\mathscr{H}_3} - Q) = \mathbb{1}_{\mathscr{H}_3} - P \wedge Q$$

but

$$P_n \wedge Q_n = \mathbb{1}_{\mathscr{H}_3} - [(\mathbb{1}_{\mathscr{H}_3} - P_n) \vee (\mathbb{1}_{\mathscr{H}_3} - Q_n)]$$

hence

$$P_n \wedge Q_n \to P \wedge Q$$

Remark. $\mathscr{L}(\mathscr{H}, \mathbb{D})$ is *not* a topological lattice. Indeed, given any bidimensional subspace $\mathscr{H}_2$ of $\mathscr{H}$ we have

$$x, x_n \in \mathscr{H}_2, \qquad x_n \to x \Rightarrow P_{x_n} \overset{W}{\to} P_x$$

If $x_n \neq x \; \forall_n$, then $P_{x_n} \vee P_x = \mathbb{1}_{\mathscr{H}_2}, \forall n$; therefore, $P_x \vee P_x$ does not converge to $P_x \vee P_x = P_x$.

4. *The 'Topology of States'*

Let $\mathscr{L}$ be a logic and $\mathscr{P}$ the set of all pure states of $\mathscr{L}$. We introduce in $\mathscr{L}$ a topology which we call the 'topology of states'.

Given a net $\{a_\alpha\}_{\alpha\in A}$ in $\mathscr{L}$ we say that $\{a_\alpha\}$ 'converges' to $a \in \mathscr{L}$ if, *for every* $s \in \mathscr{P}$, the net $\{s(a_\alpha)\}$ converges to $s(a)$ in the usual topology of $\mathbb{R}$.

This 'convergence' satisfies conditions (n_1)–(n_4) quoted in Appendix D. In fact:

(n_1) $a_\alpha = a \;\forall\alpha\in A \Rightarrow s(a_\alpha) = s(a) \;\forall\alpha \in A, \forall s \in \mathscr{P} \Rightarrow s(a_\alpha) \to s(a) \;\forall s \in \mathscr{P}$ and thus, by definition, $a_\alpha \to a$;

(n_2) if $\{a_\beta\}_{\beta\in B}$ is a subnet of $\{a_\alpha\}_{\alpha\in A}$, $\{s(a_\beta)\}$ is a subnet of $\{s(a_\alpha)\}$ for every $s \in \mathscr{P}$; if $a_\alpha \to a$, $s(a_\alpha) \to s(a_\alpha) \;\forall s \in \mathscr{P}$, thus the subnet $\{s(a_\beta)\}$ converges to $s(a)$ for every $s \in \mathscr{P}$ and therefore, by definition, $a_\beta \to a$;

(n_3) if $a_\alpha \nrightarrow a$ there exists an $\hat{s} \in \mathscr{P}$ such that $\hat{s}(a_\alpha) \nrightarrow \hat{s}(a)$; therefore there exists a subnet $\{\hat{s}_\beta\}$ of the net $\{\hat{s}(a_\alpha)\}$ no subnet of which converges to $\hat{s}(a)$; we can now find a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ such that $\hat{s}(a_\beta) = \hat{s}_\beta$; obviously, $a_\beta \nrightarrow a$ and no subnet of $\{a_\beta\}$ can converge to a;

(n_4) if $\{a_\alpha\}_{\alpha\in A}$ converges to a and, for each $\alpha \in A$, $\{a_{\alpha,\beta}\}_{\beta\in B\alpha}$ converges to a_α, for every $s \in \mathscr{P}$, $s(a_\alpha) \to s(a)$ and $s(a_{\alpha,\beta}) \to s(a_\alpha)$; but the converging nets in $\mathbb{R}$ satisfy the law of iterated limits, thus $s(a_{\alpha,\omega(\alpha)}) \to s(a) \;\forall s \in \mathscr{P}$; therefore, by definition $a_{\alpha,\omega(\alpha)} \to a$.

We can now introduce a topology in $\mathscr{L}$, the convergence relative to which is equivalent to the above convergence, taking as the family of closed sets the family of the subsets S of $\mathscr{L}$ such that: $\{x_\alpha\} \subset S$ and $x_\alpha \to x$ imply $x \in S$.

This topology will be called the 'topology of states'.

In a logic $\mathscr{L}$ the topology of states is a T_1-topology iff the set of pure states is separating. In fact, suppose $\mathscr{P}$ is separating and consider the net $\{a_\alpha\}_{\alpha\in A}$ with $a_\alpha = a$, $\forall\alpha \in A$, then $a_\alpha \to a$ and $a_\alpha \to b$ imply $s(a) = s(b) \;\forall s \in \mathscr{P}$, whence $a = b$. Conversely, suppose $\mathscr{P}$ is not separating; then there exist two distinct elements a and b of $\mathscr{L}$ such that $s(a) = s(b) \;\forall s \in \mathscr{P}$, hence the net $\{a_\alpha\}_{\alpha\in A}$ with $a_\alpha = a \;\forall\alpha \in A$ converges to a and b, and therefore a is not closed.

Furthermore, it must be remarked that the topology of states is a Hausdorff topology if it is a T_1-topology. For if $a_\alpha \to a$ and $a_\alpha \to b$ then $s(a) = s(b) \;\forall s \in \mathscr{P}$, and this implies $a = b$ if $\mathscr{P}$ is separating.

It is obvious that once the topology of states is introduced in $\mathscr{L}$ every pure state is a continuous mapping. We can prove also that the orthocomplementation is a continuous mapping. Since, for every $a \in \mathscr{L}$ and every $s \in \mathscr{P}$, $s(a) + s(a^\perp) = 1$, from $a_\alpha \to a$ we have $\forall s \in \mathscr{P}\; s(a_\alpha{}^\perp) \to s(a^\perp)$, and therefore $a_\alpha{}^\perp \to a^\perp$.

An interesting remark is that if $\mathscr{L}$ is a 'classical logic' (Varadarajan, 1968, Chapter 1), $\mathscr{L}$ with the topology of states is a *topological lattice*. The logic $\mathscr{L}$ of a classical system is the σ-algebra of Borel subsets M of the phase space Ω of the system and the set of pure states of $\mathscr{L}$ is the set $\{\chi_\omega, \omega \in \Omega\}$ where, for each $M \in \mathscr{L}$,

$$\chi_\omega(M) =: \begin{cases} 1 & \text{if } \omega \in M \\ 0 & \text{if } \omega \notin M \end{cases}$$

Now, suppose $M_\alpha \to M$ and $N_\alpha \to N$; then we can find for each $\omega \in \Omega$ and $\alpha_0 \in A$ such that $\alpha \geqslant \alpha_0$ implies

$$\chi_\omega(M_\alpha) = \begin{cases} 1 & \text{if } \omega \in M \\ 0 & \text{if } \omega \notin M \end{cases}$$

and

$$\chi_\omega(N_\alpha) = \begin{cases} 1 & \text{if } \omega \in N \\ 0 & \text{if } \omega \notin N \end{cases}$$

Therefore, for $\alpha \geqslant \alpha_0$, $\chi_\omega(M_\alpha \wedge N_\alpha) = 1$ iff $\omega \in M \wedge N$ and $\chi_\omega(M_\alpha^\perp \wedge N_\alpha)$ iff $\omega \notin M \wedge N$.

This means that $\forall \omega \in \Omega$, $\chi_\omega(M_\alpha \wedge N_\alpha) \to \chi_\omega(M \wedge N)$ and thus, by definition, $M_\alpha \wedge N_\alpha \to M \wedge N$. Moreover,

$$M_\alpha \vee N_\alpha = (M_\alpha{}^\perp \wedge N_\alpha{}^\perp) \overset{\perp}{\to} (M^\perp \vee N^\perp)^\perp = M \wedge N$$

Theorem 4.1. In the logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ the topology of states is the topology induced by the weak operator topology of $\mathscr{B}(\mathscr{H})$.

Proof. For each normalised vector $u \in \mathscr{H}$ let us consider the mapping s_u: $\mathscr{L}(\mathscr{H}, \mathbb{D}) \to \mathbb{R}$ defined by $s_u(P) =: \|Pu\|^2$. From the Gleason theorem (Varadarajan, 1968, Theorem 7.23) we know that the mapping s_u is a pure state of the logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ for every normalised $u \in \mathscr{H}$ and that for every pure state s of $\mathscr{L}(\mathscr{H}, \mathbb{D})$ there exists a normalised $u \in \mathscr{H}$ such that $s = s_u$. Therefore, if $\{P_\alpha\}$ is a net in $\mathscr{L}(\mathscr{H}, \mathbb{D})$, $P_\alpha \to P \in \mathscr{L}(\mathscr{H}, \mathbb{D})$ means $\|P_\alpha u\|^2 \to \|Pu\|^2$ for every normalised $u \in \mathscr{H}$. But this is equivalent to $(P_\alpha x, x) \to (Px, x)$ $\forall x \in \mathscr{H}$, which is equivalent to $(P_\alpha x, y) \to (Px, y)$ $\forall x, y \in \mathscr{H}$, by the polarisation principle.

Remark 4.1. While a 'classical logic' with the topology of states is a topological lattice, a 'quantum logic' with the topology of states is not.

Remark 4.2. The topology of states has a clear physical meaning: two 'propositions' are 'near' if the probability of the result—yes—is almost the same for the two propositions in all the pure states; that is, if the two propositions give nearly the same information on the physical system.

5. *Conditions under which a Logic with the Topology of States Determines* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{Q}$

Let $\mathscr{L}$ be a projective logic such that every family of mutually orthogonal points is at most countable. We assume that $\mathscr{L}$ satisfies the following topological conditions:

($\mathscr{L}_1$) in $\mathscr{L}$ is introduced the topology of states, and $s(a) = s(b)$ $\forall s \in \mathscr{P}$ implies $a = b$ (that is, $\mathscr{P}$ is separating);

($\mathscr{L}_2$) for every finite element a of $\mathscr{L}$, $\mathscr{L}[0, a]$ is a compact subset of $\mathscr{L}$ with the topology of states;

($\mathscr{L}_3$) $\mathscr{L}$ with the topology of states is second countable;
($\mathscr{L}_4$) for every line l of $\mathscr{L}$ with the topology of states the set of all points of l, but one arbitrarily chosen, is a connected set;
($\mathscr{L}_5$) no plane of $\mathscr{L}$ is trivial; for every plane u of $\mathscr{L}$ with the topology of states the intersection point of two lines in u is a continuous function of the two lines and the union line of two points in u is a continuous function of the two points.

Obviously, all these topological conditions for $\mathscr{L}$ are conditions on the set of pure states of $\mathscr{L}$. From the condition ($\mathscr{L}_1$) it follows immediately that $\mathscr{L}$ with the topology of states results in a Hausdorff space.

Moreover, as we can see from Theorem 4.1 and from Section 3, the projective logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ of all the projections in a separable Hilbert space $\mathscr{H}$ over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with the weak operator topology, satisfies all the above conditions.

Theorem 5.1. $\mathscr{L}$ is isomorphic to the projective logic $\mathscr{L}(V, \langle .,. \rangle)$ of all $\langle .,. \rangle$-closed linear manifolds of V, where V is a vector space over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$).

Proof. let $\tilde{\mathscr{L}}$ be the generalised geometry consisting of all the finite elements of $\mathscr{L}$. Following Varadarajan's notations, for every line l of $\tilde{\mathscr{L}}$ we denote by $\beta_2(l)$ the set of all the projectivities of l onto itself which can be written as a product of two perspectivities.

Let us choose arbitrarily a plane u in $\tilde{\mathscr{L}}$ and a line l in u.

On l we fix three distinct points O ('origin'), E ('unit point'), W ('point at infinity').

Let $\mathbb{D}$ be the set of all the points of l, but W. We introduce in $\mathbb{D}$ the following two operations:

$$A + B =: p_B A$$

$$A \cdot B =: \begin{cases} q_B A & \text{if } B \neq 0 \\ O & \text{if } B = 0 \end{cases}$$

where p_B is the unique special projectivity of $\beta_2(l)$ with W as its fixed point and such that $p_B O = B$ and q_B is the unique general projectivity of $\beta_2(l)$ with O and W as its first and second fixed points and such that $q_B E = B$ (see Varadarajan, 1968, Chapter 2). With these operations $\mathbb{D}$ becomes a division ring (in general not commutative) with O and E as its zero and unit respectively.

The generalised geometry $\tilde{\mathscr{L}}$ results in it being isomorphic to the generalised geometry of all the finite dimensional linear manifolds of a linear space V over a division ring which is isomorphic to the division ring $\mathbb{D}$ constructed as above, no matter how the plane u of $\tilde{\mathscr{L}}$, the line l in u and the three points O, E, W in l are chosen (Varadarajan, 1968, Chapter 5).

We want now to prove that $\mathbb{D}$ is a topological division ring which is locally compact, second countable and connected.

(a) $\mathbb{D}$ is a topological division ring. Given u, l, O, E, W as above, let $n \neq l$ and $r \neq l$ be two distinct lines lying on u and containing W, and let X be a point on r distinct from W (Fig. 1) (we recall that there exists at least three distinct lines containing a given point).

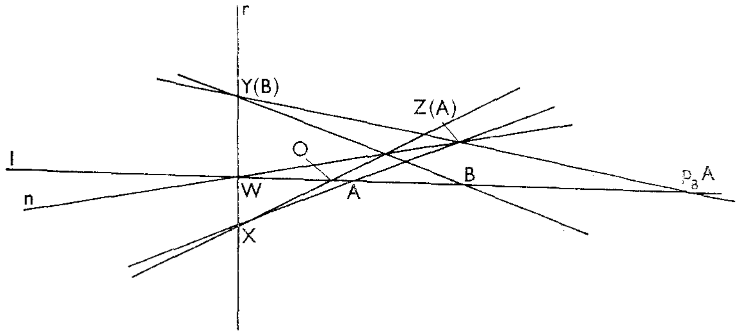

FIG. 1.

For each point B of l let $Y(B) =: (((X \vee O) \wedge n) \vee B) \wedge r$ and for each point A of l distinct from W let $Z(A) =: (X \vee A) \wedge n$. Since $Z(A)$ and $Y(B)$ are continuous functions of A and B respectively, $f(A,B) =: (Z(A) \vee Y(B)) \wedge l$ is a continuous function of the pair (A,B). But $f(A,B) = P_B A$; therefore, $A + B$ is a continuous function of (A,B).

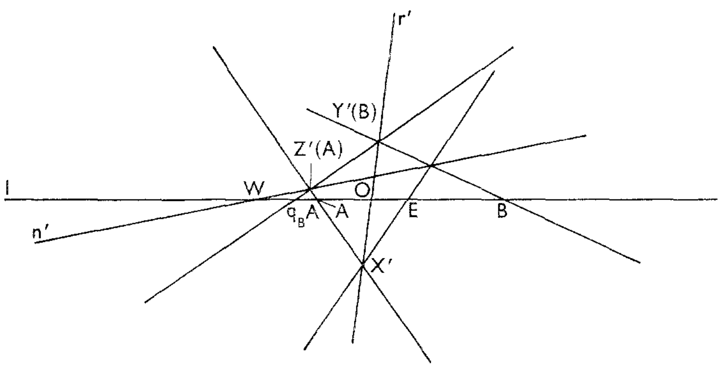

FIG. 2,

Let now $n' \neq l$ be a line in u containing W, $r' \neq l$ aline on u containing O, and let X' be a point on r' distinct from O (Fig. 2). For each point B of l let $Y'(B) =: (((X' \vee E) \wedge n') \vee B) \wedge r'$ and for each point A of l let $Z'(A) =:$ $(X' \vee A) \wedge n'$. Since $Z'(A)$ and $Y'(B)$ are continuous functions of A and B

respectively, $g(A,B) =: (Z'(A) \vee Y'(B)) \wedge I$ is a continuous function of the pair (A,B). But

$$g(A,B) = \begin{cases} q_B A & \text{if } B \neq 0 \\ 0 & \text{if } B = 0 \end{cases}$$

therefore $A.B$ is a continuous function of (A,B).

(b) $\mathbb{D}$ is second countable. This follows immediately from $(\mathscr{L}_3)$, since second countability is a hereditary property.

(c) $\mathbb{D}$ is connected. This follows immediately from $(\mathscr{L}_4)$.

(d) $\mathbb{D}$ is locally compact. By condition $(\mathscr{L}_2)$ $\mathscr{L}[\mathbf{0},I]$ is compact. The set $C =: \{\mathbf{0}, I, W\}$, is closed because we deal with a T_1-topology. Now, the division ring $\mathbb{D}$ is the complementary set of C in $\mathscr{L}[\mathbf{0},I]$ and $\mathscr{L}$ is a Hausdorff space; therefore $\mathbb{D}$ is locally compact.

It is well known that a second countable connected and locally compact division ring is isomorphic and homeomorphic to one of the topological division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ (Pontrjagin, 1946, Theorem 45). Thus we have obtained that $\tilde{\mathscr{L}}$ is isomorphic to the generalised geometry of all finite dimensional linear manifolds of a linear space V over a division ring $\mathbb{D}$ which is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$. The statement of the theorem follows from Varadarajan (1968, Theorem 7.40).

We have thus proved that if a projective logic $\mathscr{L}$ satisfies the conditions given at the beginning of the section, then $\mathscr{L}$ determines a division ring $\mathbb{D}$ which is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$.

To go further on and establish an isomorphism with the projective logic of all the closed linear manifolds of a separable Hilbert space, the θ-bilinear form $\langle .,. \rangle$ must have the properties of an inner product. When so, V with the inner product $\langle .,. \rangle$ is complete, that is a Hilbert space (Varadaragan, 1968, Theorem 7.42).

Now if $\mathbb{D}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, the θ-bilinear form $\langle .,. \rangle$ is an inner product iff the antiautomorphism θ is the identity, the complex conjugation or the canonical conjugation respectively. If $\mathbb{D}$ is $\mathbb{R}$ or $\mathbb{Q}$ the antiautomorphism θ determined by $\mathscr{L}$ is the identity or the canonical conjugation respectively. If $\mathbb{D}$ is $\mathbb{C}$ the antiautomorphism θ is the complex conjugation iff it is continuous (see Varadarajan, 1968, p. 179).

We can then arrive at the projective logic $\mathscr{L}(\mathscr{H}, \mathbb{D})$ if we require that the antiautomorphism θ is continuous. For what we have just said this is a restriction only in the case $\mathbb{D} = \mathbb{C}$.

Therefore, if we take into account Theorems 4.1 and 5.1 and the results of Section 3 we can state the following theorem.

Theorem 5.2. Let $\mathscr{L}$ be any logic. Then

(1) if $\mathscr{L}$ is a projective logic with the properties: (a) every family of mutually orthogonal points of $\mathscr{L}$ is at most countable, (b) conditions $(\mathscr{L}_1)$–$(\mathscr{L}_5)$ are satisfied, then $\mathscr{L}$ is isomorphic to the projective logic $\mathscr{L}(V, \langle .,. \rangle)$ of all linear manifolds closed relative to the θ-bilinear form $\langle .,. \rangle$, where V is a linear space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$;

(2) if, in addition, the antiautomorphism θ is continuous, then V is a separable Hilbert space with $\langle .,. \rangle$ as inner product.

Conversely, if $\mathscr{L}$ is isomorphic to $\mathscr{L}(\mathscr{H}, \mathbb{D})$ where $\mathscr{H}$ is a separable Hilbert space over $\mathbb{D}$ ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$) with $\dim \mathscr{H} \geqslant 4$, then $\mathscr{L}$ is a projective logic with the properties: (a) every family of mutually orthogonal points of $\mathscr{L}$ is at most countable, (b) conditions $(\mathscr{L}_1)$–$(\mathscr{L}_5)$ are satisfied, (c) the antiautomorphism θ is continuous.

As a conclusion, two remarks are in order.

First, the continuity of the antiautomorphism θ of $\mathbb{D}$ followed by the general topological assumptions on $\mathscr{L}$ could have been expected. We have not yet settled this question.

Second, if in theorem 5.2 we want to cut out the possibility $\mathbb{D} = \mathbb{Q}$, we must require that in $\mathscr{L}$ the 'Pappus property' holds. Conversely, if we want $\mathbb{D} = \mathbb{Q}$, we must require that the 'Pappus property' does not hold. Indeed, the validity of the 'Pappus property' is a necessary and sufficient condition for the commutativity of $\mathbb{D}$ (Artin, 1962, p. 73).

Acknowledgement

We wish to thank Dr. F. Gallone for many helpful discussions on the subject.

Appendix

A. Lattices

(1) *Definitions and notations*

A *lattice* is a poset (with the order relation denoted by $\leqslant$) in which for every pair of elements a, b there exists $\sup(a,b)$ and $\inf(a,b)$.

A lattice $\mathscr{L}$ is called *σ-complete* if $\sup_i\{a_i\}$ and $\inf\{a_i\}$ exists for every countable set $\{a_i\}$ of elements of $\mathscr{L}$; it is called *complete* if $\sup M$ and $\inf M$ exists for every subset M of $\mathscr{L}$.

We use the notations $a \vee b$, $a \wedge b$, $\bigvee_i \{a_i\}$, $\bigwedge_i \{a_i\}$, $\bigvee M$ and $\bigwedge M$ for $\sup(a,b)$, $\inf(a,b)$, $\sup_i\{a_i\}$, $\inf_i\{a_i\}$, $\sup M$ and $\inf M$ respectively.

If a lattice admits a greatest element and a least element (this is always the case for a complete lattice) we call them **I** and **0** respectively.

A subset B of a lattice $\mathscr{L}$ such that $\forall a \in \mathscr{L}$, $a \neq \mathbf{0}$, $\exists\, B_{(a)} \subset B \,|\, a = \bigvee B_{(a)}$ is called a *base* of $\mathscr{L}$.

(2) *Distributive, modular and weakly modular lattices*

A lattice is called *distributive* if

(D.1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $\quad \forall a, b, c \in \mathscr{L}$;

(D.2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee v)$, $\quad \forall a, b, c \in \mathscr{L}$;

modular if

(M) $a \leqslant b \Rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b$, $\quad \forall c \in \mathscr{L}$;

weakly modular if

(WM) $a \leqslant b, \quad c \leqslant a^{\perp} \Rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b.$

Obviously, (D.1) $\Leftrightarrow$ (D.2), (D.2) $\Rightarrow$ (M) and (M) $\Rightarrow$ (WM).

(3) *Complemented and orthocomplemented lattice*

A lattice $\mathscr{L}$ with $\mathbf{0}$ and $\mathbf{I}$ is called complemented if for every $a \in \mathscr{L}$ there exists in $\mathscr{L}$ at least an a' such that

$$a \wedge a' = \mathbf{0}, \qquad a \vee a' = \mathbf{I}$$

An orthocomplementation in a lattice $\mathscr{L}$ with $\mathbf{0}$ and $\mathbf{I}$ is a mapping $\perp$: $\mathscr{L} \to \mathscr{L}$ such that

(1) $a \wedge a^{\perp} = \mathbf{0}, \quad \forall a \in \mathscr{L};$
(2) $a \vee a^{\perp} = \mathbf{I}, \quad \forall a \in \mathscr{L};$
(3) $a \leqslant b \Rightarrow b^{\perp} \leqslant a^{\perp}, \quad \forall a, b \in \mathscr{L};$
(4) $a^{\perp\perp} = a, \quad \forall a \in \mathscr{L}.$

In a given lattice $\mathscr{L}$ an orthocomplementation doesn't need to exist and, if there is one, it doesn't need to be the unique one. A lattice $\mathscr{L}$ with $\mathbf{0}$ and $\mathbf{I}$ and with a given orthocomplementation is called an *orthocomplemented lattice*.

A complemented distributive lattice is usually called a *Boolean lattice*. In a Boolean lattice the complement a' is unique for every a, it is an orthocomplement and the unique one.

(4) *Atomic lattices*

An element c of a lattice $\mathscr{L}$ with $\mathbf{0}$ is called an *atom* (or a *point*) of $\mathscr{L}$ if $c \neq 0$ and $a \leqslant c \Rightarrow a = \mathbf{0}$ or $a = c$.

A lattice $\mathscr{L}$ is called *atomic* if it has a base consisting of atoms (and therefore of all the atoms of $\mathscr{L}$).

(5) *Topological lattices*

A topological lattice is a set $\mathscr{L}$ endowed with compatible lattice and topological structures, the compatibility meaning that $\vee$ and $\wedge$ are continuous mapping from $\mathscr{L} \times \mathscr{L}$ (with the product topology) in $\mathscr{L}$; if $\mathscr{L}$ as a lattice is orthocomplemented, the continuity of the orthocomplementation is also required.

B. GEOMETRIES (Vanadarajan, 1968)

(1) *Dimension function*

Given a complemented lattice $\mathscr{L}$, a *dimension function* on $\mathscr{L}$ is a mapping d: $\mathscr{L} \to \mathbb{R}$ such that

(1) $d(a) \geqslant 0, \quad \forall a \in \mathscr{L}, \quad d(\mathbf{0}) = 0;$
(2) $a \leqslant b, \quad a \neq b \Rightarrow d(a) < d(b);$
(3) $d(a \vee b) + d(a \wedge b) = d(a) + d(b), \quad \forall a, b \in \mathscr{L}.$

A complemented lattice with a dimension function is modular (Varadarajan, 1968, Lemma 2.2).

We recall that a *chain* in a poset $\mathscr{P}$ is a subset of $\mathscr{P}$ for which the induced ordering is a total ordering. The *length* of a chain is the number of its elements. A lattice $\mathscr{L}$ is said to have a finite length if there is an integer κ such that the length of any chain in $\mathscr{L}$ does not exceed κ. The least upper bound of the lengths of $\mathscr{L}$ is called the *length* of $\mathscr{L}$.

A complemented modular lattice $\mathscr{L}$ of finite length is atomic and there exists a unique dimension function d on $\mathscr{L}$ such that $d(c) = 1$ for every atom c of $\mathscr{L}$ (Varadarajan, 1968, Theorem 2.8). This dimension function is called *canonical.*

(2) *Geometries*

The centre of a complemented modular lattice $\mathscr{L}$ is the set of elements of $\mathscr{L}$ having a unique complement. If the centre is the set $\{\mathbf{0},\mathbf{I}\}$, $\mathscr{L}$ is called *irreducible.*

A *geometry* is a complemented, modular, irreducible lattice of finite length.

In a geometry $\mathscr{L}$ an element l will be called a *line* of $\mathscr{L}$ if $l = p_1 \vee p_2$, where p_1 and p_2 are two distinct points (atoms) of $\mathscr{L}$, an element u will be called a *plane* of $\mathscr{L}$ is $u = l \vee p$, where l is a line and p a point not lying (not contained) on l (a lies on b (a is contained in b) means $a \leqslant b$).

Every line contains at least three distinct points (Varadarajan, 1968, Theorem 2.15).

A plane u is said to be *trivial* if every line lying in u contains exactly three points.

If we consider in a geometry $\mathscr{L}$ the canonical dimension d we have

$$d(a) = 1 \Leftrightarrow a \text{ is a point of } \mathscr{L}$$
$$d(a) = 2 \Leftrightarrow a \text{ is a line of } \mathscr{L}$$
$$d(a) = 3 \Leftrightarrow a \text{ is a plane of } \mathscr{L}$$

In this paper for the dimension function in a geometry $\mathscr{L}$ we shall always mean the canonical dimension function. We define the dimension of a geometry $\mathscr{L}$ as the canonical dimension of its geatest element and write $\dim \mathscr{L}$ for it (note that $\dim \mathscr{L}$ equals length of $\mathscr{L}$ minus 1).

(3) *Generalised geometries*

In a lattice $\mathscr{L}$ we denote by $\mathscr{L}[a_1, a_2]$ the set of elements $b \in \mathscr{L}$ such that $a_1 \leqslant b \leqslant a_2$ with the induced order relation.

A *generalized geometry* is a lattice $\mathscr{L}$ with $\mathbf{0}$ such that for every $a \in \mathscr{L}, a \neq \mathbf{0}$: $\mathscr{L}[\mathbf{0}, a]$ is a geometry.

Obviously if a generalised geometry has also $\mathbf{I}$, it is a geometry.

Since for every $a \in \mathscr{L}$, $a \neq \mathbf{0}$, $\mathscr{L}[\mathbf{0}, a]$ is a geometry, we can introduce in $\mathscr{L}$ a dimension function simply defining $\dim(\mathbf{0}) = 0$, $\dim(a) =: \dim \mathscr{L}[\mathbf{0}, a]$. This dimension function has the properties

(1) $\dim(a) \geqslant 0, \qquad \forall a \in \mathscr{L}, \qquad \dim(\mathbf{0}) = 0;$
(2) $a \leqslant b, a \neq b \Rightarrow \dim(a) \angle \dim(b)$:

(3) $\dim(a \vee b) + \dim(a \wedge b) = \dim(a) + \dim(b)$:
(4) $\dim(a) = 1$, $\quad a$ is a point of $\mathscr{L}$.

Again, an element a of a generalised geometry $\mathscr{L}$ with $\dim(a) = 2$ will be called a line of $\mathscr{L}$ and an element a of $\mathscr{L}$ with $\dim(a) = 3$ a plane of $\mathscr{L}$.

Let now a_1, a_2 be two elements of a generalised geometry $\mathscr{L}$ and suppose that

$$\dim(a_1) = \dim(a_2) = r > 0$$
$$\dim(a_1 \wedge a_2) = r - 1$$

Then there exists at least one point $P \leqslant a_1 \vee a_2$ lying neither on a_1 nor on a_2; for every such a point P, $P \vee a_1 = P \vee a_2$ and the mapping

$$\eta: \mathscr{L}[\mathbf{0}, a_1] \to \mathscr{L}[\mathbf{0}, a_2], \qquad \eta(b) =: (x \vee b) \wedge a_2$$

is an isomorphism of $\mathscr{L}[\mathbf{0}, a_1]$ onto $\mathscr{L}[\mathbf{0}, a_2]$; moreover, $\eta(b) = b$ for every $b \leqslant a_1 \wedge a_2$ (Varadarajan, 1968, Lemma 5.3).

The mapping η is called a *perspectivity of a_1 on a_2 with centre P*. Given two lines l_1 and l_2 in a plane u a projectivity of l_1 onto l_2 is a one–one mapping from l_1 onto l_2 which is a product of perspectivities.

C. θ-BILINEAR FORMS (Varadarajan, 1968)

An antiautomorphism θ of a division ring $\mathbb{K}$ is an invertible mapping $\lambda \rightsquigarrow \lambda^\theta$ of $\mathbb{K}$ onto itself such that

(1) $(\lambda + \mu)^\theta = \lambda^\theta + \mu^\theta, \qquad \forall \lambda, \mu \in \mathbb{K}$;
(2) $(\lambda\mu)^\theta = \mu^\theta \lambda^\theta, \qquad \forall \lambda, \mu \in \mathbb{K}$.

Obviously, an antiautomorphism of $\mathbb{K}$ is an automorphism iff $\mathbb{K}$ is commutative.

An antiautomorphism θ of $\mathbb{K}$ is called involutive if $(\lambda^\theta)^\theta = \lambda \; \forall \lambda \in \mathbb{K}$.

If $\mathbb{K}$ is a division ring, θ an antiautomorphism of $\mathbb{K}$ and V a linear space over $\mathbb{K}$†, a *θ-bilinear form* on VxV is a mapping $\langle .,. \rangle : VxV \to \mathbb{K}$ such that

$$(1) \begin{cases} \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle, & \forall x_1, x_2, y \in V \\ \langle x, y_1 + y_2 \rangle = \langle x_1, y_1 \rangle + \langle x, y_2 \rangle, & \forall x_1\, y_1, y_2 \in V \end{cases}$$

(2) $\langle \lambda x, \mu y \rangle = \lambda \langle x, y \rangle \mu^\theta, \qquad \forall x, y \in V, \qquad \forall \lambda, \mu \in \mathbb{K}$.

A θ-bilinear form is called *symmetric* if

$$\langle x, y \rangle = \langle y, x \rangle^\theta, \qquad \forall x, y \in \Lambda$$

non-singular if

$$\begin{cases} \langle x, y \rangle = 0, & \forall y \in V \Rightarrow x = 0 \\ \langle x, y \rangle = 0 & \forall x \in V \Rightarrow y = 0 \end{cases}$$

definite if

$$\begin{cases} \langle x, x \rangle = 0 \Rightarrow x = 0 \\ \langle z, z \rangle = 1 \quad \text{for some } z \in \Lambda \end{cases}$$

† By a linear space over $\mathbb{K}$ we understand, to be definite, a left linear space over $\mathbb{K}$.

A definite θ-bilinear form is obviously non-singular.

For every definite symmetric θ-bilinear form the antiautomorphism θ is involutive.

D. NETS (Pontrjagin, 1968)

A *directed set* is a set A with a directing relation, that is with a binary relation $\leqslant$ such that:

(1) $\alpha \leqslant \alpha, \qquad \forall \alpha \in A$;
(2) $\alpha, \beta, \gamma \in A, \qquad \alpha \leqslant \beta, \qquad \beta \leqslant \gamma \Rightarrow \alpha \leqslant \gamma$;
(3) if $\alpha, \beta \in A$ there is a $\gamma \in A$ such that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$.†

Given a family $\{A_\sigma\}_{\rho \in \Sigma}$ of directed sets the *product directed set* $\pi_\sigma A_\sigma$ is the cartesian product of the sets A_σ (that is, the set of all the functions $\omega: \sum \to \bigcup_\sigma A_\sigma$ such that $\omega(\sigma) \in A_\sigma$ for every $\sigma \in \Sigma$) with the directing relation defined by: $\omega \leqslant \omega'$ if $\omega(\sigma)^\sigma \leqslant \omega'(\sigma)$, $\forall \sigma \in \Sigma$.

Given a set X, any mapping from any directed set to X is called a *net* in X. For a net in X we will use the notation $\{x_\alpha\}_{\alpha \in A}$ or simply $\{x_\alpha\}$.

Given two nets $\{x_\alpha\}_{\alpha \in A}$ and $\{y_B\}_{\beta \in B}$ in X, the net $\{y_\beta\}_{\beta \in B}$ is called a *subnet* of the net $\{x_\alpha\}_{\alpha \in A}$ if there exists a mappping $\pi: B \to A$ such that

(1) $y_\beta = x_{\pi(\beta)}, \qquad \forall \beta \in B$,
(2) $\forall \alpha_0 \in A$ there is a $\beta_0 \leqslant B$ such that $\beta \geqslant \beta_0 \Rightarrow \pi(\beta) \geqslant \alpha_0$.

Let now X be a topological space. A net $\{x_\alpha\}_{\alpha \in A}$ in X is said to converge to an element $x \in X$ (in symbols, $x_\alpha \to X$) if for every neighbourhood V of x there exists an $\hat{\alpha}_U \in A$ such that $\alpha \geqslant \hat{\alpha}_U \Rightarrow x_\alpha \in V$. If $x_\alpha \to X$, X is called a limit of $\{x_\alpha\}_{\alpha \in A}$.

It can be shown that a subset N of X is closed in X iff for every net $\{x_\alpha\}_{\alpha \in A}$ in N all the limits of $\{x_\alpha\}_{\alpha \in A}$ belong to N and that a mapping f from X in a topological space Y in continuous at $x_0 \in X$ iff for every net $\{x_\alpha\}$ in X which converges to x_0 the net $\{f(x_\alpha)\}$ converges to x_0.

The above defined convergence of nets in a topological space X has the following properties:

(n_1) for each $x \in X$; $\qquad x_\alpha = X \, \forall \alpha \in A \Rightarrow x_\alpha \to X$;
(n_2) if $x_\alpha \to X$ and $\{x_\beta\}$ is a subnet of $\{x_\alpha\}$ then $x_\beta \to X$;
(n_3) if $\{x_\alpha\}$ does not converge to x there exists a subnet of $\{x_\alpha\}$ no subnet of which converges to x.
(n_4) (law of iterated limits) let A be a directed set, B_α a directed set for each $\alpha \in A$ and C the product directed set $A \times \prod_\alpha B_\alpha$; if the net $\{x_\alpha\}_{\alpha \in A}$ converges to x and if, for each fixed $\alpha \in A$, the net $\{x_{\alpha,\beta}\}_{\beta \in B\alpha}$ converges to x_α, then the net $\{x_{\alpha,\omega(\alpha)}\}_{(\alpha,\omega) \in C}$ converges to x.

Suppose now we have a set X and a family e of pairs, each pair being formed by a net $\{x_\alpha\}$ in X and a point $x \in X$. For a pair $(\{x_\alpha\}, x)$ belonging to e we say that $\{x_\alpha\}$ e-converges to x. A family e is called a *convergence class for X*

† It is immaterial for the theory of nets whether or not it is assumed that $\alpha \geqslant \alpha'$ and $\alpha' \geqslant \alpha$ imply $\alpha = \alpha'$.

if it satisfies (n_1)–(n_4) (obviously, with e-convergence substituted for convergence).

It is a fundamental result the fact that, given a set X and a convergence class e for X, a topology can be introduced in X the convergence relative to which is equivalent to the e-convergence. One simply takes as the family of closed sets the family of the subsets S of X such that: if $\{x_\alpha\} \subset S$ and $\{x_\alpha\}$ e-converges to x, then $x \in S$.

The topology introduced in X in this way is a T_1 topology iff

(T_1) for each $x \in X$, $x_\alpha = x \; \forall \alpha \in A$ and $\{x_\alpha\}$ e-converges to y imply $x = y$;

it is a T_2 topology (Hausdorff topology) iff

(T_2) $\{x_\alpha\}$ e-converges to x and $\{x_\alpha\}$ e-converges to y imply $x = y$.

References

Artin, E. (1962). *Algèbre Géométrique*. Gauthier-Villars, Paris.

Dixmier, J. (1969). *Les Algèbres d'Operateurs dans l'Espace Hilbertien*. Gauthier-Villars, Paris.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. Addison Wesley. (1968)

Kato, T. (1966). *Perturbation Theory of Linear Operators*. Springer-Verlag, Berlin.

Kelley, J. L. (1965). *General Topology*. Van Nostrand Co., New York.

Piron, C. (1964). Axiomatique Quantique, Thése, Université de Lausanne, Faculté des Sciences, Bale, Imprimerie Birkhauser S.A.

Pontrjagin, L. (1946). *Topological Groups*. Princeton University Press, Princeton.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. 1. Van Nostrand Co., Princeton, New Jersey.